Extension of Lord-Wingersky Algorithm to Computing Test Score Distributions for Polytomous Items

Bradley A. Hanson

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Assume there are K polytomously scored items and let U_k be a random variable representing the score on item k. There are n_k possible categories of response for item k for which unique scores are assigned, so U_k can take the values $s_{k1} < s_{k2} < \ldots < s_{kn_k}$. Let X_k be the sum of the item scores for the first k items:

$$X_k = \sum_{j=1}^{k} U_j \, .$$

The distribution of X_k for $k = 2, \ldots, K$ can be written as

$$\begin{aligned} \Pr(X_k = z) &= \Pr(X_{k-1} + U_k = z) \\ &= \sum_{u=s_{k1}}^{s_{kn_k}} \sum_{x:x+u=z} \Pr(X_{k-1} = x, U_k = u) \, . \end{aligned} \tag{1}$$

Let $x' = x + u$, then Equation 1 can be written as

$$\begin{aligned} \Pr(X_k = z) &= \Pr(X_{k-1} + U_k = z) \\ &= \sum_{u=s_{k1}}^{s_{kn_k}} \sum_{x':x'=z} \Pr(X_{k-1} = x' - u, U_k = u) \\ &= \sum_{u=s_{k1}}^{s_{kn_k}} \Pr(X_{k-1} = z - u, U_k = u) \, . \end{aligned} \tag{2}$$

Let Θ be a random variable such that U_k, $k = 1, \ldots, K$, are mutually independent given Θ. Using Equation 2, the conditional distribution of X_k given Θ for $k = 2, \ldots, K$ can be written as

$$\begin{aligned} \Pr(X_k = z \mid \Theta = \theta) &= \Pr(X_{k-1} + U_k = z \mid \Theta = \theta) \\ &= \sum_{u=s_{k1}}^{s_{kn_k}} \Pr(X_{k-1} = z - u, U_k = u \mid \Theta = \theta) \\ &= \sum_{u=s_{k1}}^{s_{kn_k}} \Pr(X_{k-1} = z - u \mid \Theta = \theta) \Pr(U_k = u \mid \Theta = \theta) \, , \end{aligned} \tag{3}$$

since X_{k-1} and U_k are independent given Θ.

If $n_k = 2$, with $s_{k1} = 0$ and $s_{k2} = 1$ for all k, then Equation 3 is the formula used by Lord and Wingersky (1984, Equation 4) for computing the conditional distribution of X_K given Θ recursively. Equation 3 gives a generalization of the formula used by Lord and Wingersky (1984) to the case where item scoring is polytomous (or dichotomous with item scores different from 0 and 1).

Reference

Lord, F. M., & Wingersky, M. S. (1984). Comparison of IRT true-score and equipercentile observed-score "equatings." *Applied Psychological Measurement*, *8*, 453-461.